MA 26500 Midterm Exam 2 Spring 2021

Name: Solutions

ID number: ______________________

Instructions:

1. You have 50 minutes to complete this exam.
2. There are 10 problems on this exam. Eight are multiple choice and two of them are free-response problems.
3. **Circle** one **and only one** option for each multiple-choice problem. No partial credit will be given for multiple-choice problems.
4. Show **all** relevant work on free-response problems. Partial credit will be given for clear steps leading to solutions. **Little to no credit will be given for little to no work.**
5. No books, notes, or calculators are allowed.
6. Please turn off your cell phone.

Purdue University faculty and students commit themselves towards maintaining a culture of academic integrity and honesty. The students taking this exam are not allowed to seek or obtain any kind of help from anyone to answer questions on this test. If you have questions, consult only an instructor or a proctor. You are not allowed to look at the exam of another student. You may not compare answers with anyone else or consult another student until after you finish your exam and hand it in to a proctor or to an instructor. You may not consult notes, books, calculators, cameras, or any kind of communications devices until after you finish your exam and hand it in to a proctor or to an instructor. If you violate these instructions you will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe and may include an F in the course. All cases of academic dishonesty will be reported to the Office of the Dean of Students. Your instructor and proctors will do everything they can to stop and prevent academic dishonesty during this exam. If you see someone breaking these rules during the exam, please report it to the proctor or to your instructor immediately. Reports after the fact are not very helpful.

I agree to abide by the instructions above:

Signature: ______________________

1. (10 points) Circle either True (**T**) or False (**F**) for the following statements:

$0 \neq \det A = \det(A - 0I)$

T / **F** : If an $n \times n$ matrix is invertible, then 0 is not one of its eigenvalues.

T / **F** : Let A be a square matrix. If A is diagonalizable, then A^k is diagonalizable for any integer $k \geq 1$. If $A = PDP^{-1} \Rightarrow A^k = PD^kP^{-1}$ diagonal

T / **F** : The map $T : M_{2\times 2} \to \mathbb{R}^1$ given by $T(A) = \det A$ is a linear transformation. $\det(A+B) \neq \det A + \det B$

T / **F** : If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.

T / **F** : If an $n \times n$ matrix is diagonalizable, then it has n distinct eigenvalues.

I_n is diagonalizable but $\lambda = 1$ is its only eigenvalue

2. (8 points) Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 & 5 \\ 0 & 1 & 4 & -6 & 7 \\ 0 & 0 & -2 & 1 & 4 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Which of the following statements are true for the matrix A?

✓ (i) A has four distinct eigenvalues. $1, -2, 3, 4$

✓ (ii) 1 is an eigenvalue of A of multiplicity 2. char. poly has $(x-1)^2$ as a factor

✗ (iii) -2 is an eigenvalue whose associated eigenspace has dimension 2.

A. (i) only

B. (ii) only

C. (ii) and (iii) only

D. (i) and (ii) only

E. (i) and (iii) only

F. (i), (ii), and (iii)

$\dim E_{-2} \leq \text{mult } {-2} = 1$

Or $A - (-2)I$ has only 1 non-pivot column

3. (9 points) Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be the linear transformation given by $T(p(t)) = \begin{bmatrix} p(0) \\ p(0) \end{bmatrix}$, where $\mathbb{P}_2$ is the vector space of polynomials with real coefficients of degree at most 2. Find the dimension of the kernel of T and the dimension of the range of T.

A. $\dim(\ker(T)) = 2$, $\dim(\text{range}(T)) = 1$

B. $\dim(\ker(T)) = 2$, $\dim(\text{range}(T)) = 2$

C. $\dim(\ker(T)) = 1$, $\dim(\text{range}(T)) = 2$

D. $\dim(\ker(T)) = 3$, $\dim(\text{range}(T)) = 1$

E. $\dim(\ker(T)) = 2$, $\dim(\text{range}(T)) = 0$

F. $\dim(\ker(T)) = 0$, $\dim(\text{range}(T)) = 1$

$p(t) = at^2 + bt + c$

$T(p(t)) = \begin{bmatrix} c \\ c \end{bmatrix}$

ker T

$\begin{bmatrix} c \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow c = 0$

$p(t) = at^2 + bt$

$\Rightarrow \{t^2, t\}$ a basis

$\dim(\ker T) = 2$

Range T

$\begin{bmatrix} c \\ c \end{bmatrix} = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ a basis

$\dim(\text{Range } T) = 1$

4. (9 points) Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$. If $A = PDP^{-1}$ for invertible P and diagonal matrix D, which of the following could be D?

A. $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

D. $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

E. $\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$

F. $\begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{bmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix}$$

$$= (2-\lambda)(1-\lambda) - 12$$

$$= \lambda^2 - 3\lambda + 2 - 12$$

$$= \lambda^2 - 3\lambda - 10$$

$$= (\lambda-5)(\lambda+2)$$

$\lambda = 5, -2$ eigenvalues

5. (9 points) Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation given by $T(p(t)) = p'(t) + p(t)$. Which of the following is the matrix representation of T with respect to the standard basis $\beta = \{1, t, t^2\}$?

A. $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

F. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$T(1) = 0 + 1 = 1 \qquad [T(1)]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$T(t) = 1 + t \qquad [T(t)]_\beta = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$T(t^2) = 2t + t^2 \qquad [T(t^2)]_\beta = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$$[T]_\beta = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

6. (9 points) Which **one** of the following statements is **false** for the matrix $\begin{bmatrix} 11 & 10 \\ 1 & 2 \end{bmatrix}$?

A. A has exactly 2 eigenvalues ✓

B. There exists a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = -\mathbf{x}$ ✗ -1 is not an eigenvalue!

C. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector for A ✓ $\begin{bmatrix} 11 & 10 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

D. There exist 2 linearly independent eigenvectors for A ✓ A is diagonalizable

E. $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ is an eigenvector for A ✓ $\begin{bmatrix} 11 & 10 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

F. There exists a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = 12\mathbf{x}$ ✓ 12 is an eigenvalue

$$\det(A - \lambda I) = \det\begin{bmatrix} 11-\lambda & 10 \\ 1 & 2-\lambda \end{bmatrix}$$
$$= (11-\lambda)(2-\lambda) - 10$$
$$= \lambda^2 - 13\lambda + 12$$
$$= (\lambda-1)(\lambda-12)$$

eigenvalues

$\lambda = 1, 12$

7. (9 points) Which **one** of the following sets is **not** a subspace of $M_{2\times2}$, where $M_{2\times2}$ is the vector space of 2×2 matrices with real entries?

A. the set of all matrices of the form

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix},$$

where a, b and d are real numbers.

B. $\text{Span}\left\{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

C. $\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$

D. The set of 2×2 matrices with real entries and determinant 0 — Not closed under addition!

E. The set of all 2×2 diagonal matrices with real entries

F. The set of all 2×2 lower triangular matrices with real entries

$\det\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$ but $\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$

$\det\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0$

8. (9 points) Given that $\lambda = 5$ is an eigenvalue of multiplicity 2 for

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix},$$

which of the following is a basis of eigenvectors for the eigenspace associated to $\lambda = 5$?

A. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

F. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$$A - 5I = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

x_2 and x_3 free

$x_1 + 2x_3 = 0$

$\Rightarrow x_1 = -2x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

9. (14 points) Suppose $x'(t) = A \cdot x(t)$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, and $x'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$

(a) Find a fundamental set of solutions for the system of differential equations.

(b) Find the unique solution $x(t)$ with $x(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Eigenvalues: $\lambda = 1, 2$ (since A upper triangular)

$\underline{\lambda = 1}$

$$A - I = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(or any scalar multiple)

$\underline{\lambda = 2}$

$$A - 2I = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

a) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} \right\}$ a fundamental set of solutions.

b) $x(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$

$$x(0) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3 \end{array}\right] \Rightarrow \begin{cases} c_1 = -1 \\ c_2 = 3 \end{cases}$$

$x(t) = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$ is the unique solution.

10. (14 points) Diagonalize the following matrix over the complex numbers if possible. In other words find diagonal matrix D and invertible matrix P such that $A = PDP^{-1}$. **You do not need to compute P^{-1}!**

$$A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & -2 \\ 2 & -\lambda \end{bmatrix} = \lambda^2 + 4 \overset{?}{=} 0$$

$\lambda = \pm 2i$ eigenvalues

$\underline{\lambda_1 = 2i}$

$$A - 2iI = \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \sim \begin{bmatrix} i & 1 \\ 2 & -2i \end{bmatrix} \sim \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} \qquad v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$(R_2 = -2i\, R_1)$ (or any scalar multiple)

$\lambda_2 = -2i = \overline{\lambda_1}$ so we can take $v_2 = \overline{v_1} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1}$$